

课程: 5.4 几类特殊函数的积分

题号	一	二	三	四	五	六	七	八	九	十	总分
得分											
阅卷人											

考生注意: 答题时应注明题号, 并按题号顺序答题。

题号	答题内容
5.4节 P102	廖国勇
1	$\int \frac{1}{x^2-2x-3} dx = \frac{1}{4} \int (\frac{1}{x-3} - \frac{1}{x+1}) dx = \frac{1}{4} \ln \frac{x-3}{x+1} + C$
2	$\int \frac{1}{x^2+2x+5} dx = \int \frac{1}{(x+1)^2+2^2} d(x+1) = \frac{1}{2} \arctan \frac{x+1}{2} + C$
3	$\int \frac{1}{1+\sqrt{x}} dx = \int \frac{2\sqrt{x}}{1+\sqrt{x}} d\sqrt{x} = \int (2 - \frac{2}{1+\sqrt{x}}) d\sqrt{x} = 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$
4	$\int \frac{1}{1+\sin x} dx = \int \frac{1}{(\cos \frac{x}{2} + \sin \frac{x}{2})^2} dx = 2 \int \frac{1}{[\sqrt{2} \sin(\frac{x}{2} + \frac{\pi}{4})]^2} d(\frac{x}{2} + \frac{\pi}{4})$ $= \int \csc^2(\frac{x}{2} + \frac{\pi}{4}) d(\frac{x}{2} + \frac{\pi}{4})$ $= -\cot(\frac{x}{2} + \frac{\pi}{4}) + C$ $-\cot(\frac{x}{2} + \frac{\pi}{4}) = -\frac{\cos(\frac{x}{2} + \frac{\pi}{4})}{\sin(\frac{x}{2} + \frac{\pi}{4})} = -\frac{\cos(\frac{x}{2})\cos\frac{\pi}{4} - \sin(\frac{x}{2})\sin\frac{\pi}{4}}{\sin(\frac{x}{2})\cos\frac{\pi}{4} + \cos(\frac{x}{2})\sin\frac{\pi}{4}}$ $= -\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\sin\frac{x}{2} + \cos\frac{x}{2}} = \frac{(\sin\frac{x}{2} - \cos\frac{x}{2})}{(\sin\frac{x}{2} + \cos\frac{x}{2})} \cdot \frac{\sin\frac{x}{2} - \cos\frac{x}{2}}{\sin\frac{x}{2} - \cos\frac{x}{2}}$ $= \frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2} - \cos^2\frac{x}{2}} = \frac{1 - \sin x}{-\cos x} = \tan x - \sec x$ 另解, $\int \frac{1}{1+\sin x} dx = \int \frac{1-\sin x}{(1+\sin x)(1-\sin x)} dx = \int (\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}) dx$ $= \int (\sec^2 x - \sec x \cdot \tan x) dx = \tan x - \sec x + C$ 注, 不定积分结果形式可以多种, 但实质是一样.

题号

P102

二 1

求不定积分 $\int \frac{x}{(x^2+1)(x-2)} dx$

解 $\frac{x}{(x^2+1)(x-1)} = -\frac{2}{5} \frac{x}{x^2+1} + \frac{1}{5} \frac{1}{x^2+1} + \frac{2}{5} \frac{1}{x-2}$

原式 $= \int (-\frac{2}{5} \frac{x}{1+x^2} + \frac{1}{5} \frac{1}{1+x^2} + \frac{2}{5} \frac{1}{x-2}) dx$
 $= -\frac{1}{5} \ln(1+x^2) + \frac{1}{5} \arctan x + \frac{2}{5} \ln|x-2| + C$

2

求不定积分 $\int \frac{1}{(x+1)^2(x^2+1)} dx$

解: 设 $\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}$
 $= \frac{A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2}{(x+1)^2(x^2+1)}$

$\Rightarrow 1 = A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2 \quad (1)$

把 $x=0, x=-1, x=i, x=-i$ 代入 (1) 得

$$\begin{cases} 1 = A + B + D \\ 1 = 2B \\ 1 = (C+i+D)(2i) = -2C + 2Di \\ 1 = (-Ci+D)(-2i) = -2C - 2Di \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = \frac{1}{2} \\ C = -\frac{1}{2} \\ D = 0 \end{cases}$$

原式 $= \frac{1}{2} \int \frac{1}{(x+1)} dx + \frac{1}{2} \int \frac{1}{(x+1)^2} dx - \frac{1}{2} \int \frac{x}{1+x^2} dx$
 $= \frac{1}{2} \ln|1+x| - \frac{1}{2} \frac{1}{(1+x)} - \frac{1}{4} \ln(1+x^2) + C$

题号	答 题 内 容
P103	廖国勇
二 3	求不定积分 $\int \frac{1}{x(1+x^9)} dx$
	解: 原式 $= \int \frac{1}{x^9(1+x^9)} x^8 dx$ $= \int (\frac{1}{x^9} - \frac{1}{1+x^9}) \frac{1}{9} dx^9$ $= \frac{1}{9} \int \frac{1}{x^9} dx^9 - \frac{1}{9} \int \frac{1}{1+x^9} d(1+x^9)$ $= \frac{1}{9} \ln x^9 - \frac{1}{9} \ln 1+x^9 + C$ $= \frac{1}{9} \ln \frac{x^9}{1+x^9} + C$
4	求不定积分 $\int \frac{1}{\sqrt{1-x} + \sqrt{1+x}} dx$
	解: 设 $t = \sqrt{1-x}$, 则 $x = 1-t^2$, $dx = -2t dt$ 原式 $= \int \frac{1}{t^3+t^2} (-2t^5) dt$ $= \int \frac{-2t^3}{1+t} dt = \int (\frac{-2t^3-2}{1+t} + \frac{2}{1+t}) dt$ $= -2 \int (t^2-t+1) dt + 2 \int \frac{1}{1+t} dt$ $= -2(\frac{1}{3}t^3 - \frac{1}{2}t^2 + t) + 2\ln 1+t + C$ $= -\frac{2}{3}\sqrt{1-x}^3 + \sqrt{1-x}^2 - 2\sqrt{1-x} + 2\ln(1+\sqrt{1-x}) + C$

题号

P103

二 5

求不定积分 $\int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}} dx$

解: 设 $\sqrt[3]{\frac{x-1}{x+1}} = t$, 则 $x = \frac{1+t^3}{1-t^3}$ $dx = \frac{6t^2}{(1-t^3)^2} dt$

$$\begin{aligned} \int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}} dx &= \int \frac{1}{(x+1)(x-1) \sqrt[3]{\frac{x-1}{x+1}}} dx \\ &= \int \frac{1}{\left(\frac{1+t^3}{1-t^3}+1\right)\left(\frac{1+t^3}{1-t^3}-1\right) \cdot t} \cdot \frac{6t^2}{(1-t^3)^2} dt \end{aligned}$$

$$= \int \frac{3}{2} \frac{1}{t^2} dt$$

$$= -\frac{3}{2} \frac{1}{t} + C$$

$$= -\frac{3}{2} \sqrt[3]{\frac{x+1}{x-1}} + C$$

6 求不定积分 $\int \frac{1}{1+\sin x + \cos x} dx$

解: 原式 $= \int \frac{1}{(1+\cos x) + \sin x} dx$

$$= \int \frac{1}{2\cos^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= \int \frac{1}{1+\tan \frac{x}{2}} (\sec \frac{x}{2} d\frac{x}{2})$$

$$= \int \frac{1}{1+\tan \frac{x}{2}} d\tan \frac{x}{2}$$

$$= \ln |\tan \frac{x}{2} + 1| + C$$

$$= \ln \left| \frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{\cos \frac{x}{2}} \right| + C$$

题号

P104

答 题 内 容

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二 7 求不定积分 $\int \frac{1}{3+\sin^2 x} dx$ 解: $\int \frac{1}{3+\sin^2 x} dx$

$$= \int \frac{1}{3\sin^2 x + 3\cos^2 x + \sin^2 x} dx$$

$$= \int \frac{1}{4\sin^2 x + 3\cos^2 x} dx$$

$$= \int \frac{1}{4\tan^2 x + 3} \cdot \frac{1}{\cos^2 x} dx$$

$$= \int \frac{1}{4\tan^2 x + 3} d\tan x$$

$$= \frac{1}{2} \int \frac{1}{(2\tan x)^2 + (\sqrt{3})^2} d(2\tan x)$$

$$= \frac{1}{2\sqrt{3}} \arctan \frac{2\tan x}{\sqrt{3}} + C$$

注 $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C \quad (a>0)$

题号

答 题 内 容

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二 8

$$\int \frac{\sin x}{\sin x + \cos x} dx$$

$$\text{解: } (\sin x + \cos x)' = \cos x - \sin x$$

$$\sin x = \frac{1}{2} [(\sin x + \cos x) - (\sin x + \cos x)']$$

于是

$$\int \frac{\sin x}{\sin x + \cos x} dx = \frac{1}{2} \int \left[\frac{\sin x + \cos x}{\sin x + \cos x} - \frac{(\sin x + \cos x)'}{\sin x + \cos x} \right] dx$$

$$= \frac{1}{2} \int dx - \frac{1}{2} \int \frac{1}{\sin x + \cos x} d(\sin x + \cos x)$$

$$= \frac{1}{2} x - \frac{1}{2} \ln |\sin x + \cos x| + C$$

5.2节 P₉₂第9题 P₉₃第16题与本题一样解法更一般地, $\int \frac{a \sin x + b \cos x}{\sin x + \cos x} dx$, 可以先设

$$a \sin x + b \cos x = c(\sin x + \cos x) + d(\sin x + \cos x)'$$

$$= (c-d) \sin x + (c+d) \cos x$$

$$\text{即 } \begin{cases} a = c-d \\ b = c+d \end{cases} \Rightarrow \begin{cases} c = \frac{a+b}{2} \\ d = \frac{b-a}{2} \end{cases}$$

于是

$$\int \frac{a \sin x + b \cos x}{\sin x + \cos x} dx = \int c \frac{\sin x + \cos x}{\sin x + \cos x} dx + \int d \frac{1}{\sin x + \cos x} d(\sin x + \cos x)$$

$$= cx + d \ln |\sin x + \cos x| + C$$

$$= \frac{a+b}{2} x + \frac{b-a}{2} \ln |\sin x + \cos x| + C$$

类似地可以解决 $\int \frac{c \sin x + d \cos x}{a \sin x + b \cos x} dx$, 例如

$$\int \frac{\sin x + 2 \cos x}{3 \sin x + 4 \cos x} dx$$

$$\sin x + 2 \cos x = e(3 \sin x + 4 \cos x) + f(3 \sin x + 4 \cos x)'$$

 e, f 为待定的常数